A general mathematical structure for the time-reversal operator.

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The aim of this work is the mathematical analysis of the physical time-reversal operator and its definition as a geometrical structure**,** in such a way that it could be generalized to the purely mathematical realm. Rigorously, only having such a "time-reversal structure" it can be decided whether a dynamical system is time-symmetric or not. The "time-reversal structures" of several important physical and mathematical examples are presented, showing that there are some mathematical categories whose objects are the (classical or abstract) "time-reversal systems" and whose morphisms generalize the Wigner transformation.

I. INTRODUCTION.

The dynamics and the thermodynamics of both, classical and quantum physical systems, are modelized by the mathematical theory of classical and abstract dynamical systems. It is obvious that the *physical* notion of "time-symmetric (or asymmetric) dynamical systems" requires the definition of a "time-reversal operator", K [17]. In fact, every known model of a physical dynamical system *has* some K operator. E.g., the dynamic of classical physical systems is described in the cotangent fiber bundle $T^*(N)$ of its configuration manifold N, and therefore the action of $K : T^*(N) \rightarrow T^*(N)$ is defined as

$$p_q \mapsto -p_q \tag{1}$$

for any linear functional p on $q \in N$, or in coordinate's language:

$$(q^i, p_i) \mapsto (q^i, -p_i) \tag{2}$$

in a particular (q^i, p_i) coordinate system.

In Quantum Mechanics there is the well known Wigner antiunitary operator K defined through the complex conjugation in the position (wave functions) representation:

$$\psi(x,t) \mapsto \psi(x,-t)^* \tag{3}$$

In the last few years it was demonstrated that ordinary Quantum Mechanics (with no superselection sectors) can be naturally included in the Hamiltonian formalism of its real Kählerian differentiable manifold of quantum states [6] [7] [8]. The latter one is the real but infinite dimensional manifold of the associated projective space $\mathbf{P}(\mathcal{H})$ of its Hilbert states space $\mathcal{H}$ [1]. From this point of view, $K : \mathbf{P}(\mathcal{H}) \rightarrow \mathbf{P}(\mathcal{H})$ acts as the cannonical projection to the quotient of (3)

$$[\psi(x,t)] \mapsto [\psi(x,-t)^*] \tag{4}$$

Moreover, this result has been generalized to more general quantum systems through its characteristic C*-algebra A, and its pure quantum states space $\partial K(A)$ turns out to be a projective Kähler bundle over the spectrum $\widehat{A}$, whose fiber over the class of a state ψ, is isomorphic to $\mathbf{P}(\mathcal{H}_\psi)$, being $\mathcal{H}_\psi$ its GNS (Gelfand-Naimark-Segall) representation's space [9]. More recently these authors have relaxed the Kählerian structure, showing "minimal" mathematical structures involved in the quantum principles of superposition and uncertainty, with the aim of considering non linear extensions of quantum mechanics [10].

[1]We should remember the fact that ordinary pure quantum states are not *vectors* ψ (normalized or not) of a Hilbert space $\mathcal{H}$, but rays or *projective equivalence classes of vectors* $[\psi] \in \mathbf{P}(\mathcal{H})$.

But, what happens in more general dynamical systems? Some of them -such as the Bernouilli systems and certain Kolmogorov-systems [2]- are purely mathematical. Nevertheless, the notion of time-symmetry seems to make sense also for them. So, it would be interesting to know what kind of mathematical structures are involved in these systems.

Our aim is to show that:

1.-*The mere existence of the time reversal operator is a mathematical structure* consisting of a non trivial involution K of the states space of a general system (with holonomic constraints) M, which splits into a K-invariant submanifold (coordinatized by q^i) and whose complementary set (the field of the effective action of K, coordinatized by p_i) is a manifold with the same dimension of M. This structure is logically independent of the symplectic one [1], which doesn't require such a splitting at all. In fact, the essence of symplectic geometry, as its own etymology shows, is the "common enveloping" of q and p, loosing any "privilege" between them. Actually, *this K-structure is defined by the action of that part of the complete Galilei group -including the time-reversal- which is allowed to act on the phase space manifold M by the constraints.* In fact, only on the homogeneous Euclidean phase space $M' = \mathbb{R}^{6n}$, it is possible to have the transitive action [14] of the complete Galilei group.

2.-*It is possible to define generalized and purely mathematical "time-reversals"* allowing a generalization of the notion of "time-symmetry" for a wider class of dynamical systems, including all Bernouilli systems. In fact there are mathematical categories whose objects are the time-reversal (classical or abstract) systems (M, K) and whose morphisms generalize the Wigner transformation [25].

3.-When the states space has additional structures, *there is a possibility of getting a richer time-reversal compatible with these structures.* For example, in Classical Mechanics the canonical K is a symplectomorphism of phase space, and the Wigner quantum operator is compatible with the Kählerian structure of $\mathbf{P}(\mathcal{H})$. At first sight (2) is quite similar to (3) and it seems to be some kind of "complex conjugation" (and the even dimensionality of phase space reinforces this idea). We will prove this fact, namely, the existence of an almost complex structure J with respect to which K is an almost complex time-reversal. This increases the analogy with Quantum Mechanics, where the strong version of the Heisemberg Uncertainty Principle, [7] [8] is equivalent to the existence of a complex structure J, by means of which the Wigner time-reversal is defined.

4.-*It is possible to make a definition of time-reversal systems so general* that it includes among its examples the real line, the Minkowski space-time, the cotangent fiber bundles, the quantum systems, the classical densities function space, the quantum densities operator space, the Bernouilli systems, etc.

The paper is organized as follows: In section 2, the general theory of **reversals** and **time-reversal systems** is developed. In section 3, the theory of **abstract reversals** and **abstract time-reversal systems** is given. In section 4, many important examples of time-reversal systems are given. In section 5, we give the abstract time-reversal of Bernouilli systems and we show explicitly the geometrical meaning of our definitions for the Baker's transformation.

II. REVERSALS AND TIME-REVERSAL SYSTEMS

Definition: Let M be a real paracompact, connected, finite or infinite-dimensional differentiable manifold, and let $K : M \to M$ be a diffeomorphism. We will say that K is a **reversal** on M, and that (M, K) is a **reversal system** if the following conditions are satisfied:

(r.1) K is an involution, i.e. $K^2 = I_M$

(r.2) The set N of all fixed points of K is an immersed submanifold of M, such that $M - N$ is a connected or disconnected submanifold of the same dimension of M. (In particular, this implies that M is a non trivial involution, i.e. $K \neq I_M$, the identity function on M)

We will say that N is the **invariant submanifold** of the reversal system.

Definition: Let (M, K) be a reversal system. We will say that M is **K-orientable** if $M - N$ is composed of two diffeomorphic connected components M_+ y M_-, and if

$$K(M_+) = M_- \text{ , and } K(M_-) = M_+ \tag{5}$$

M is **K-oriented** when -conventionally or arbitrarily- one of these components is selected as **"positively oriented"**. In that case, K changes the K-orientation of M.

If there is a complex (or almost complex) structure J on M (and therefore $J' = -J$ is another one) and if, in addition, K satisfies:

(c.r.) K is complex (or almost complex) as a map from (M, J) to $(M, J') = (M, -J)$, i.e. :

$$K_* \circ J = -J \circ K_* \tag{6}$$

we will say that K is a **complex (or almost complex) reversal, or a conjugation.**

Similarly, if a symplectic (or almost symplectic) 2-form ω is given on M [2] (and therefore $\omega' = -\omega$ is another one) and if, in addition to (r.1) y (r.2), K satisfies:

(s.r.) K is a symplectomorphism from (M, ω) to $(M, \omega') = (M, -\omega)$, i.e. :

$$K^* \omega = -\omega \tag{7}$$

we will say that K is a **symplectic (or almost symplectic) reversal.** If we have a symplectic (or almost symplectic) reversal system (M, ω, K), then for every

$$m \in M : K_* : T_m(M) \to T_m(M)$$

is a (toplinear) isomorphism, and if

$$i : N \to M \text{ is the immersion} : i(q) = m$$

and $i_*(X_q) = X_{i(q)}$ is the induced isomorphism, we can define an almost complex structure J:

$$\begin{aligned} J\left(X_{i(q)}\right) &:= Y_m \Leftrightarrow \omega\left(X_{i(q)}, Y_m\right) = 1 \\ J\left(Y_m\right) &:= -X_{i(q)} \end{aligned} \tag{8}$$

that is to say, by defining the pairs of "conjugate" vectors (and extending by linearity). Then

$$K_*\left(X_{i(q)}\right) = X_{i(q)} \ , \ K_*\left(Y_m\right) = -Y_m$$

When (M, ω, J, g) is a Kähler (or almost Kähler) manifold, and K satisfies the properties (r.1), (r.2), (c.r.) and (s.r.), we will say that K is a **Kählerian (or almost Kählerian) reversal.** In that case, K is also an isometry

$$K^* g = g \tag{9}$$

with respect to the Kähler metric g defined by:

$$g(X, Y) = -\omega(X, JY) \text{ for all vector fields } X \text{ and } Y. \tag{10}$$

Definition: Let (M, K) be a reversal system such that there is a class $\mathcal{F}$ of flows $(S_t)_{t \in \mathbb{R}}$ or / and cascades $(S_t = S^t)_{t \in \mathbb{Z}}$ on M such that, for any $m \in M$, and any t in $\mathbb{R}$ (or in $\mathbb{Z}$) satisfies:

$$(K \circ S_t \circ K)(m) = S_{-t}(m) \tag{11}$$

Then we will say that K is a **time-reversal** for $\mathcal{F}$ on M. (In Physics we can take $\mathcal{F}$ as a class of physical interest. For example, in Classical Mechanics we can take the class of all Hamiltonian flows over a fixed phase space and in Quantum Mechanics the class of solutions of the Schrödinger equation in a fixed states space, etc.)

Only having a time-reversal on M, **time-symmetric (or asymmetric)** dynamical systems (S_t) (flows $(S_t)_{t \in \mathbb{R}}$; or cascades $(S_t = S^t)_{t \in \mathbb{Z}}$) can be defined. In fact, (S_t) will be considered as **symmetric with respect to the time-reversal** K**,** if it fulfills $\forall m \in M$ the condition (11) (or **asymmetric** if it doesn't).

When M is orientable (oriented) with respect to a time-reversal K, we will say that it is **time-orientable (oriented).**

Definition: By a **morphism** of the reversal system (M, K) into (M', K'), we mean a differentiable map f of M into M' such that

$$f \circ K = K' \circ f \tag{12}$$

As the composition of two morphisms is a morphism and the identity I_M is a morphism, we get a **category of reversal systems**, whose objects are the reversal systems and whose morphisms are the morphisms of reversal systems. Also we have the subcategories of symplectic, almost complex, Kählerian, etc. reversal systems.

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[2] In the infinite-dimensional case we require ω to be *strongly non-degenerate* [10] in the sense that the map $X \mapsto \omega(X, .)$ is a toplinear isomorphism.

III. ABSTRACT REVERSALS AND ABSTRACT TIME-REVERSAL SYSTEMS

Definition: Let (M, μ), be a measure space, and let $K : M \rightarrow M$ be an isomorphism (mod 0) [2]. We will say that K is an **abstract reversal** on (M, μ) and that (M, μ, K) is an **abstract reversal system** if the following conditions are satisfied:

(a.r.1) K is an involution, i.e. $K^2 = I_M$ (mod 0)

(a.r.2) The set N of all fixed points of K is a measurable subset of null measure of M, and so $\mu[M - N] = \mu[M]$ (In particular, this implies that M is a non trivial involution, i.e. $K \neq I_M$, the identity function on M)

We will say that N is the **invariant subset** of the abstract reversal system.

Definition: Let (M, μ, K) be an abstract reversal system such that there is a class $\mathcal{F}$ of measure preserving flows $(S_t)_{t\in\mathbb{R}}$ or / and cascades $(S_t = S^t)_{t\in\mathbb{Z}}$ on M such that, for all $m \in M$, and all t in $\mathbb{R}$ (or in $\mathbb{Z}$) it satisfies (11). Then, we will say that K is a **time reversal** for $\mathcal{F}$ on (M, μ).

Only having an abstract time-reversal on M, **time-symmetric (or asymmetric)** abstract dynamical systems (S_t) (flows $(S_t)_{t\in\mathbb{R}}$; or cascades $(S_t = S^t)_{t\in\mathbb{Z}}$) can be defined. In fact, (S_t) will be regarded as **symmetric with respect to the time-reversal** K**,** if it fulfills $\forall m \in M$ the condition (11) (or **asymmetric** if it doesn't)

Definition: By a **morphism** of the abstract reversal system (M, μ, K) into (M', μ', K'), we mean a measurable map f of (M, μ) into (M', μ') such that, $\forall A' \subset M'$ measurable:

$$\mu\left(f^{-1}(A')\right) = \mu'\left(A'\right) \text{ mod } 0 \tag{13}$$

and

$$f \circ K = K' \circ f \tag{14}$$

As the composition of two morphisms is a morphism and the identity I_M is a morphism, we get a **category of abstract reversal systems**, whose objects are the abstract reversal systems and whose morphisms are the morphisms of abstract reversal systems.

IV. EXAMPLES OF TIME-REVERSAL SYSTEMS

We will see how the mathematical structure just described can be implemented in all the classical and quantum physical systems, and also generalized to more abstract purely mathematical dynamical systems, as the Bernouilli systems.

A. The real line

Let us consider in the real line $\mathbb{R}$, the mapping $K : \mathbb{R} \rightarrow \mathbb{R}$ defined by:

$$K(t) = -t \tag{15}$$

Clearly, $\mathbb{R}$ is K-orientable, because if $N = \{0\}$, then $\mathbb{R} - \{0\} = \mathbb{R}_+ \cup \mathbb{R}_-$, and K is a canonical time-reversal for the family of translations: for $a \in \mathbb{R}$ fixed, and $t \in \mathbb{R}$,

$$S^a_t(x) := x + ta \tag{16}$$

B. The Minkowski space-time

Let $(\mathbb{R}^4, \eta)$ be the Minkowski space-time, with $\eta = \text{diag}\,(1, -1, -1, -1)$. The invariant submanifold is the spacelike hyperplane

$$N = \{(0, x, y, z) : x, y, z \in \mathbb{R}\}$$

Clearly, fixing M_+ as the halfspace containing the "forward" light cone

$$\left\{(ct, x, y, z) : c^2t^2 - x^2 - y^2 - z^2 > 0 \text{ and } t > 0\right\}$$

and M_- as the halfspace containing the "backward" light cone

$$\left\{(ct,x,y,z) : c^2t^2 - x^2 - y^2 - z^2 > 0 \text{ and } t < 0\right\}$$

and defining $K : \mathbb{R}^4 \to \mathbb{R}^4$ by:

$$K(ct,x,y,z) = (-ct,x,y,z) \tag{17}$$

we get a canonical K-orientation, equivalent to the ussual time-orientation. K is a time-reversal with respect to the temporal translations

$$S_t^A(X) := X + tA = (x^0 + ta^0, x^1, x^2, x^3) \tag{18}$$

for $A = (a^0,0,0,0) \in \mathbb{R}^4 : a^0 \neq 0$ fixed, and $t \in \mathbb{R}$.

Remark: As an effect of curvature, not every general 4-dimensional Lorentzian manifold (M,g), will be time-orientable [18]. Nevertheless, a time-orientable general space-time is necessary if we are searching for a model of a universe with an arrow of time [3] [4]. In fact, if our universe were represented by a non-time-orientable manifold, it would be impossible to define past and future in a global sense, in contradiction with all our present cosmological observations. Precisely, we know that there are no parts of the universe where the local arrow of time points differently from our own arrow.

C. The cotangent fiber bundle. Classical Mechanics.

A general cotangent fiber bundle needs not to be K-orientable. Nevertheless, we have the following result:

Theorem: The cotangent fiber bundle (of a finite dimensional differentiable manifold) $(T^*(N),\pi,N)$ [1] has a canonical almost Kählerian time-reversal (for the Hamiltonians flows on it).

Proof: Let M be the cotangent fiber bundle $T^*(N)$ of a real n-dimensional manifold N. In this case N is an embedded submanifold of M, being the embedding $i : N \to T^*(N)$ such that:

$$i(q) = 0_q \text{ (the null functional at } q)$$

Let's define

$$\begin{gathered} K \,:\, T^*(N) \to T^*(N) \\ \forall p_q \in T_q^*(N) : K(p_q) = -p_q \end{gathered} \tag{19}$$

Because of its definition, this map is obviously an involution, and by its linearity is differentiable and its differential or tangent map

$$K_* : T\left(T^*(N)\right) \to T\left(T^*(N)\right)$$

verifies:

$$K_*(X_{p_q}) = \begin{cases} X_{p_q} \text{ if } X_{p_q} \in i_*\left(T_q(N)\right) \\ -X_{p_q} \text{ if } X_{p_q} \in T_{p_q}\left(\pi^{-1}(q)\right) \end{cases} \tag{20}$$

$X_{p_q} \in T_{p_q}\left(\pi^{-1}(q)\right)$ means that it is "vertical" or tangent to the point p_q of the fibre in q. It must be taken into account that by joining a vertical base with the image of a base in N by the isomorphism i_* , we get a base of $T_{p_q}\left(T^*(N)\right)$.

Let ω be the canonical symplectic 2-form of the cotangent fiber bundle. As ω is antisymmetric, in order to evaluate $K^*\omega$, it is sufficient to consider only three possibilities:

$$\begin{aligned} &1)\ \left(X_{p_q},Y_{p_q}\right) : X_{p_q}\,, Y_{p_q} \in i_*\left(T_q(N)\right) \\ &2)\ \left(X_{p_q},Y_{p_q}\right) : X_{p_q}\,, Y_{p_q} \in T_{p_q}\left(\pi^{-1}(q)\right) \\ &3)\ \left(X_{p_q},Y_{p_q}\right) : X_{p_q} \in i_*\left(T_q(N)\right) \text{ but } Y_{p_q} \in T_{p_q}\left(\pi^{-1}(q)\right) \end{aligned}$$

In case 1)

$$\omega\left(K_*(X_{p_q}),K_*(Y_{p_q})\right) = \omega\left(X_{p_q},Y_{p_q}\right) = 0 \tag{21}$$

In case 2)

$$\omega\left(K_*(X_{p_q}), K_*(Y_{p_q})\right) = \omega\left(-X_{p_q}, -Y_{p_q}\right) = \omega\left(X_{p_q}, Y_{p_q}\right) = 0 \tag{22}$$

In case 3)

$$\begin{aligned} \omega\left(K_*(X_{p_q}), K_*(Y_{p_q})\right) &= \omega\left(X_{p_q}, -Y_{p_q}\right) \\ &= -\omega\left(X_{p_q}, Y_{p_q}\right) \end{aligned} \tag{23}$$

Thus, in any case

$$(K^*\omega)\left(X_{p_q}, Y_{p_q}\right) = \omega\left(K_*(X_{p_q}), K_*(Y_{p_q})\right) = -\omega\left(X_{p_q}, Y_{p_q}\right) \tag{24}$$

which proves that $K^*\omega = -\omega$, the (s.r.) property.

Now, let us define

$$\begin{aligned} J \,:\, T\left(T^*(N)\right) &\to T\left(T^*(N)\right) \\ J(X_{p_q}) = Y_{p_q} &\Leftrightarrow \omega\left(X_{p_q}, Y_{p_q}\right) = 1 \end{aligned} \tag{25}$$

that is to say, $J(X_{p_q})$ is the canonical conjugate of X_{p_q}.

Then, by the antisymmetry of ω, clearly $J^2 = -I$. In addition

$$\begin{aligned} \left(K_* \circ J\right)(X_{p_q}) &= K_*\left(J(X_{p_q})\right) = -J\left(X_{p_q}\right) \\ &= -J\left(K_*\left(X_{p_q}\right)\right) \\ &= \left(-J \circ K_*\right)(X_{p_q}) \end{aligned} \tag{26}$$

if $X_{p_q} \in i_*\left(T_q(N)\right)$ and

$$\begin{aligned} \left(K_* \circ J\right)(X_{p_q}) &= K_*\left(J(X_{p_q})\right) = J(X_{p_q}) \\ &= J\left(-K_*\left(X_{p_q}\right)\right) \\ &= \left(-J \circ K_*\right)(X_{p_q}) \end{aligned} \tag{27}$$

if $X_{p_q} \in T_{p_q}\left(\pi^{-1}(q)\right)$. So, K preserves both ω and J, and therefore is almost Kählerian. □

As it is well known, the phase space M of a classical system with a finite number (n) of degrees of freedom and holonomic constraints has the particular form $T^*(N)$, where N denotes the configuration space of the system. This implies the existence of a privileged submanifold N of M. We may enquire: why is this so? The answer is: because every law of Classical Mechanics is invariant with respect to the Galilei group *which contains all the spatial translations* (and it is itself a contraction of the inhomogeneous Lorentz group [13]). This forces the configuration space to be a submanifold of some *homogeneuos* $\mathbb{R}^d$ space. Now, in this submanifold we also have a privileged system of coordinates: the spatial position coordinates $q_1 = x_1, ..., q_d = x_d$, with respect to which the action of the Galilei group has its simplest affine expression. Nevertheless, in general this action will take us away from the configuration manifold N, because it doesn't fit with the constraints (think for example in the configuration space of a double pendulum -with two united threads- which is a 2-torus, contained in $\mathbb{R}^3$).

D. Classical Statistical Mechanics

Let us consider the phase space of a classical system $(T^*(N), \omega)$ and take the real Banach space $V = L^1_{\mathbb{R}}\left(T^*(N), \sigma\right)$ containing the probability densities over the phase space, where $\sigma = \omega \wedge ... \wedge \omega$ (n times) is the Liouville measure. V is a real infinite dimensional differentiable manifold modelled by itself. Then, the above defined almost Kählerian time-reversal K on $T^*(N)$ induces $\widetilde{K} : V \to V$ by:

$$\rho \mapsto \widetilde{K}(\rho) : \left(\widetilde{K}(\rho)\right)(m) := \rho\left(K(m)\right) \tag{28}$$

Clearly, $\widetilde{K}$ is a toplinear involution. Now, let us consider the set P of all ("almost everywhere" equivalent classes of) "$\widetilde{K}$-even" integrable functions

$$P = \{\rho \in V : \rho\left(K(m)\right) = \rho(m)\} \tag{29}$$

and the set I of all ("almost everywhere" equivalent classes of) the "$\widetilde{K}$-odd" integrable functions

$$I = \{\rho \in V : \rho\left(K(m)\right) = -\rho(m)\} \tag{30}$$

Trivially, $V = P \oplus I$, and there are two toplinear projectors mapping any $\rho \in V$ into its "$\widetilde{K}$-even" and "$\widetilde{K}$-odd" parts. P is the invariant subspace of $\widetilde{K}$. Its complement is the (infinite dimensional) open submanifold of ("almost everywhere" equivalent classes of) integrable functions whose "$\widetilde{K}$-odd" projection doesn't vanish.

Now, every dynamical system (S_t) (in particular those of the class $\mathcal{F}$ of K) on $T^*(N)$ induces another dynamical system (U_t) on V :

$$\left(U_t(\rho)\right)(m) := \rho\left(S_{-t}(m)\right) \tag{31}$$

Considering the class $\widetilde{\mathcal{F}}$, induced by $\mathcal{F}$, we conclude that $\widetilde{K}$ is a time-reversal.

So we have another example lacking time orientability but having a time-reversal structure .

E. Complex Banach spaces

A **complex structure** on a *real* finite (or infinite) Banach space V is a linear (toplinear) transformation J of V such that $J^2 = -I$, where I stands for the identity transformation of V. [14]

In the case of a *complex* Banach space $V_{\mathbb{C}}$, we can consider the associated real vector space (its "realification") $V = V_{\mathbb{R}}$ composed of the same set of vectors, but with $\mathbb{R}$ instead of $\mathbb{C}$, as the field of its scalars. Then, $J = iI$ is the canonical complex structure of $V_{\mathbb{R}}$.

If J is a complex structure on a finite dimensional real vector space, its dimension must be even. In any case, there exist elements X_1, X_2,..., X_n, ... of V such that

$$\{X_1, ..., X_n, ..., JX_1, ...JX_n, ...\}$$

is a basis for V [14].

Let us define $K : V \to V$ as the "conjugation", i.e. extending by linearity the assignment

$$\forall i = 1, 2, ... : K(X_i) = X_i \ ; \ K(JX_i) = -JX_i \tag{32}$$

Then the real subspace generated by $\{X_1, X_2, ...\}$, is the subspace of fixed points of K, N. So, (V, J, K) is a complex time-reversal system for the class of "non real translations" $\left(S_t^A\right)_{t\in\mathbb{R}}$ (A being a linear combination of $JX_1, ...JX_n, ...$):

$$S_t^A(X) = X + tA \ , \ t \in \mathbb{R} \tag{33}$$

In fact,

$$\begin{aligned}\left(K \circ S_t^A \circ K\right)(X) &= K\left(K(X) + tA\right) = X - tA \\ &= S_{-t}^A(X)\end{aligned} \tag{34}$$

F. Ordinary quantum mechanical systems

As a particular case of the previous example, let us consider a classical system whose phase space is $\mathbb{R}^{6n}$, and take $V = \mathcal{H} = L^2\left(\mathbb{R}^{3n}, \sigma\right)$ (actually, its realification). This choice is motivated by the fact that we want to have a Galilei-invariant Quantum Mechanics, and so *we must quantify the spatial position coordinates.* There is no cannonical or symplectic symmetry here. Only acting on the wave functions of the position coordinates the Wigner time-reversal operator will be expressed as the complex conjugation. So, we get a complex time-reversal structure.

Now, following [7], let us consider the real but infinite dimensional Kählerian manifold $\left(\mathbf{P}(\mathcal{H}), \widetilde{J}, \widetilde{\omega}, g\right)$ of the associated projective space $\mathbf{P}(\mathcal{H})$ of the Hilbert states space $\mathcal{H}$ of an ordinary quantum mechanical system. J is the complex structure of $\mathcal{H}$, and it is the local expresion of

$$\widetilde{J} : T\left(\mathbf{P}(\mathcal{H})\right) \to T\left(\mathbf{P}(\mathcal{H})\right)$$

$\left(\mathbf{P}(\mathcal{H}), \widetilde{J}, \widetilde{\omega}, g\right)$ has a canonical Kählerian time-reversal structure. In fact, we define $K : \mathcal{H} \to \mathcal{H}$ as in the previous example, and take:

$$\widetilde{K} : \mathbf{P}(\mathcal{H}) \to \mathbf{P}(\mathcal{H}) \text{ by: } \widetilde{K}\left[\psi\right] = \left[K(\psi)\right]$$

then, all the desired properties follows easily.

G. Quantum Statistical Mechanics

Let $V = L^1(\mathcal{H})$ denote the complex Banach space generated by all nuclear operators on $\mathcal{H}$ with the trace norm. This set contains the density operators of Quantum Statistical Mechanics. V is a real infinite dimensional differentiable manifold modelled by itself. Then, the above defined complex time-reversal K on $\mathcal{H}$ induces $\widehat{K} : V \to V$ by:

$$\widehat{\rho} \mapsto \widehat{K}(\widehat{\rho}) : \left(\widehat{K}(\widehat{\rho})\right)(\psi) := \widehat{\rho}\left(K(\psi)\right) \tag{35}$$

Clearly, $\widehat{K}$ is a toplinear involution. Now, let us consider the set R of all "$\widehat{K}$-real" densities

$$R = \{\widehat{\rho} \in V : \widehat{\rho}\left(K(\psi)\right) = \widehat{\rho}(\psi)\} \tag{36}$$

and the set I of all the "$\widehat{K}$-imaginary" densities

$$I = \{\widehat{\rho} \in V : \widehat{\rho}\left(K(\psi)\right) = -\widehat{\rho}(\psi)\} \tag{37}$$

Trivially, $V = R \oplus I$, and there are two toplinear projectors mapping any $\rho \in V$ into its "$\widehat{K}$-real" and "$\widehat{K}$-imaginary" parts. R is the invariant subspace of $\widehat{K}$. Its complement is the (infinite dimensional) open submanifold of ("almost everywhere" equivalent classes of) integrable functions whose "$\widehat{K}$-imaginary" projection is not null.

Now, every dynamical system (S_t) (in particular those of the class $\mathcal{F}$ of K) on $\mathcal{H}$ induces another dynamical system (U_t) on V by setting:

$$\left(U_t(\widehat{\rho})\right)(\psi) := \widehat{\rho}\left(S_{-t}(\psi)\right) \tag{38}$$

Considering the class $\widehat{\mathcal{F}}$, induced by $\mathcal{F}$, we conclude that $\widehat{K}$ is a complex time-reversal. So we have another example lacking time orientability but having a time-reversal structure.

In both Classical and Quantum Statistical Mechanics, we have used the same criterium to choose N and $\mathcal{H}$ respectively. The densities of the two theories are related by the Wigner integral W, which is an essential ingredient in the theory of the classical limit [5]. In the one dimensional case, it is the mapping $\widehat{\rho} \mapsto \rho = W\left(\widehat{\rho}\right)$ given by:

$$\rho(q,p) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \widehat{\rho}(q-\lambda, q+\lambda)\, e^{2ip\lambda}\, d\lambda \tag{39}$$

where q is the spatial position coordinate [3], p its conjugate momentum, $\rho(q,p)$ a classical density function, and

$$\begin{aligned} \widehat{\rho}(x,x') &= \left(\sum_{j=1}^{\infty} \rho_j\, \overline{\psi_j} \otimes \psi_j\right)(x,x') \\ &= \sum_{j=1}^{\infty} \rho_j\, \overline{\psi_j}(x)\psi_j(x') \end{aligned} \tag{40}$$

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[3] We want to emphasize the necessity of having an homogeneous configuration space ($\mathbb{R}$ in the one dimensional case) in order to have the translations $q \mapsto q \pm \lambda$ in W integral.

is a generic matrix element of a quantum density, being $\{\psi_j\}_{j=1}^{\infty}$ an orthonormal base of $\mathcal{H}$, $\rho_j \geqslant 0$ and $\sum\limits_{j=1}^{\infty} \rho_j = 1$.

As it is obvious by a simple change of variables,

$$W\left[\widehat{K}\left(\widehat{\rho}\right)\right](q,p) = \rho(q,-p) = \rho\left(K(q,p)\right) = \widetilde{K}\left[W\left(\widehat{\rho}\right)\right](q,p) \tag{41}$$

and therefore, W is a morphism between $\left(L^1(\mathcal{H}), \widehat{K}\right)$ and $\left(L^1_{\mathbb{R}}\left(T^*(N)\right), \widetilde{K}\right)$.

H. Koopman treatment of Kolmogorov-Systems

With the definition of time-reversal in the physical examples above, we now face the same definition in purely mathematical dynamical systems.

Let (M, μ, S_t) be a Kolmogorov system (cascade or flow). As it is well known, this implies that the induced unitary evolution U_t in $\mathcal{H} = [1]^{\perp}$ the orthogonal complement of the one dimensional subspace of the classes a. e. of the constant functions in the Hilbert space $L^2(M, \mu)$, has uniform Lebesgue spectrum of numerable constant multiplicity. This, in turn, implies the existence of a system of imprimitivity $(E_s)_{s\in\mathbb{G}}$ based on $\mathbb{G}$ for the group $(U_t)_{t\in\mathbb{G}}$, where $\mathbb{G}$ is $\mathbb{Z}$ or $\mathbb{R}$:

$$E_{s+t} = U_t E_s U_t^{-1} \tag{42}$$

Following Misra [19], we define the “Aging” operator

$$T = \int\limits_{\mathbb{G}} s\, dE_s = \begin{cases} \int\limits_{\mathbb{R}} s\, dE_s \text{ for fluxes} \\ \sum\limits_{s\in\mathbb{Z}} sE_s \text{ for cascades} \end{cases} \tag{43}$$

Then

$$U_{-t} T U_t = T + t \tag{44}$$

T is selfadjoint in the discrete case, and essentially selfadjoint in the continuous case, and there are eigenvectors in the discrete case, and generalized eigenvectors (antifunctionals) in certain riggings of $\mathcal{H}$ by a nuclear space Φ ($\Phi \prec \mathcal{H} \prec \Phi^{\times}$) in the continuous case, $(|\tau, n\rangle)_{\tau\in\mathbb{G}}$, such that:

$$T\left|\tau, n\right\rangle = \tau\left|\tau, n\right\rangle \tag{45}$$

$$U_t\left|\tau, n\right\rangle = (\tau + t)\left|\tau, n\right\rangle \tag{46}$$

Defining

$$K\left|\tau, n\right\rangle = -\left|\tau, n\right\rangle \tag{47}$$

It follows easilly that K restricted to $\mathcal{H}$ is a time-reversal for $\mathcal{F} = \{(U_t)\}$ with respect to which U_t is symmetric.

V. EXAMPLES OF ABSTRACT TIME-REVERSALS

A. Bernouilli schemes

Let M be the set $\Sigma^{\mathbb{Z}}$ of all bilateral sequences (of “bets”)

$$m = (a_j)_{j\in\mathbb{Z}} = (...a_{-2}, a_{-1}, a_0, a_1, a_2, ...) \tag{48}$$

on a finite set Σ with n elements (a “dice” with n faces). Let $\mathfrak{X}$ be the σ-algebra on M generated by all the subset of the form

$$A_j^s = \{m : a_j = s \in \Sigma\} \tag{49}$$

Clearly,

$$M = \bigcup_{s \in \Sigma} A_j^s = \bigcup_{k=1}^{n} A_j^{s_k} \tag{50}$$

Let's define a normalized measure μ on M by choosing n ordered positive real numbers $p_1, ..., p_n$ whose sum is equal to one (p_k is the "probability" of getting s_k when the "dice" is thrown), and setting:

$$\forall k : k = 1, ..., n \ : p_k = \mu(A_j^{s_k}) \tag{51}$$

$$\mu \left(A_{j_1}^{s_1} \cap ... \cap A_{j_k}^{s_k} \right) = \mu(A_{j_1}^{s_1}) ... \mu(A_{j_k}^{s_k}) \tag{52}$$

where $j_1, ..., j_k$ are all different.

Let the dynamical authomorphism S be the shift to the right:

$$\begin{aligned} S\left((a_j)_{j \in \mathbb{Z}}\right) &= (a'_j)_{j \in \mathbb{Z}} \\ \text{where:} \quad a'_j &:= a_{j-1} \end{aligned} \tag{53}$$

The shift preserves μ because

$$\mu\left(S(A_j^{s_k})\right) = \mu(A_{j+1}^{s_k}) = p_k \tag{54}$$

The above abstract dynamical scheme is called a Bernouilli scheme and denoted $B(p_1, ..., p_n)$.

Let's define a "cannonical" abstract reversal by:

$$\begin{aligned} K\left((a_j)_{j \in \mathbb{Z}}\right) &= (a'_j)_{j \in \mathbb{Z}} \\ a'_j &= a_{-j+1} \end{aligned} \tag{55}$$

Clearly, K is an isomorphism, and its invariant set

$$N = \bigcap_{j \in \mathbb{Z}} \left\{ \bigcup_{s \in \Sigma} \left(A_j^s \cap A_{-j}^s \right) \right\} \tag{56}$$

has μ-measure 0. In addition K is a time-reversal for the class $\mathcal{F}$ of all Bernouilli schemes, because

$$K \circ S \circ K = S^{-1} \tag{57}$$

being S^{-1} the shift to the left.

B. The Baker's transformation

We will show the geometrical meaning of the last two time-reversals for the Baker's transformation.

The measure space is the torus

$$M = [0,1] \times [0,1] \ / \sim = \{(x,y) \bmod 1 = [x,y] : x, y \in [0,1]\}$$

that is to say, $\sim$ is the equivalence relation that identifies the following boundary points:

$$(0,x) \sim (1,x) \text{ and } (x,0) \sim (x,1)$$

with its Lebesgue measure. The automorphism S acts as follows:

$$S(x,y) = \begin{cases} (2x, \frac{1}{2}y) & if \ 0 \leq x \leq \frac{1}{2} \ , \ 0 \leq y \leq 1 \\ (2x-1, \frac{1}{2}y + \frac{1}{2}) & if \ \frac{1}{2} \leq x \leq 1 \ , \ 0 \leq y \leq 1 \end{cases} \tag{58}$$

It's clear that S is a non-continuous but measure preserving transformation which involves a contraction in the y direction and a dilatation in the x direction: the contracting and dilating directions at every point $m \in M$ (that is, the vertical and the horizontal lines through each m).

The torus is a compact, connected Lie group and we can define an involutive automorphism K on M by putting:

$$K[x,y]=[y,x]\ ;\ x,y\in I=[0,1] \tag{59}$$

The fixed points of K constitute a submanifold of the torus: the projection of the diagonal Δ of the unit square $I\times I$

$$N=\Delta\ /\sim=\{[x,x]:x\in I\}$$

Then:

$$K\circ S^{t}\circ K=S^{-t},\ \forall t\in\mathbb{Z} \tag{60}$$

In fact, the first application of K to the generating partition of S, rotates the unit square, interchanging the x fibers with the y ones. Then by applicating S^t (that is t times S) we get a striped pattern of horizontal lines, which is rotated and yields a striped pattern of vertical lines when K is applicated again. The same pattern would be obtained if S^{-t} was used.

As it is well known [2], the Baker transformation is isomorphic to $B(\frac{1}{2},\frac{1}{2})$. In fact, the map

$$(x,y)\mapsto m=(a_j)_{j\in\mathbb{Z}}\Leftrightarrow x=\sum_{j=0}^{\infty}\frac{a_{-j}}{2^{j+1}}\text{ and }y=\sum_{j=1}^{\infty}\frac{a_j}{2^j} \tag{61}$$

is an isomorphism (mod 0). Moreover it is an isomorphism of abstract time-reversal systems, because it sends the time-reversal of (59) in the time-reversal of (55). In particular, this implies that the Baker's map is a Kolmogorov system, and therefore having the corresponding time-reversal for its Koopman treatment. These three reversals are related.

Let $\{A,B\}$ be the partition of the unit square into its left and right halves. As it is well known this partition is both independent and generating for the Baker's map. Let's define

$$\theta_0=1-\chi_A=\begin{cases}1\text{ in }A\\-1\text{ in }B\end{cases} \tag{62}$$

where χ_A is the characteristic function of the set A, as well as

$$\theta_n=U^n(\theta_0)=\theta_0\circ S^{-n}=\begin{cases}1\text{ in }S^n(A)\\-1\text{ in }S^n(B)\end{cases} \tag{63}$$

and for any finite set $F=\{n_1,...,n_F\}\subset\mathbb{Z}$, put

$$\theta_F=\theta_{n_1}...\theta_{n_F}\text{ (ordinary product of functions)} \tag{64}$$

Then, all the eigenvectors of the Aging operator T of U are of the form [20]:

$$T\theta_F=n_m\theta_F \tag{65}$$

where $n_m=\max F$. Geometrically speaking, $\rho=\theta_0$ -that we can identify with $\{A,B\}$- is an eigenvector of age 0, and if U acts n times on it we get an eigenvector of age n: θ_n -which can be identified with a set of horizontal fringes-. On the other hand if U^{-1} acts n times on it we get an eigenvector of age $-n$: θ_{-n} -which can be identified with a set of vertical fringes-. As expected, the induced action of K sends the "future" horizontal eigenstates of T to the "past" vertical ones, and reciprocally.

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